Optimizations and Lagrange Multiplier Method

Introduction

Once an objective of any real world application is well specified as a function of its control variables, which may subject to a certain number of constraint equations, we can use the Lagrange multiplier method to find the optimal solution to achieve the most desirable result. The applications of optimizations cover all domains in academies and industries.

Applications

In theoretic mathematics, we try to solve the exact equations from approximate models. But, in applied mathematics, we first make realistic assumptions that the models and measurements all have errors. Consequently, we seek for the approximate solutions that minimized their deviations from the corresponding observed data. A good start points for such approach is the Lagrange Multiplier method.

Lagrange multiplier method covers great amount of real-world applications that are to find the maximum or minimum value of a single objective function under one to multiple constraints functions.

In data mining and statistics, we wish to find the discrete probability distribution on the points with minimum Shannon information entropy. In the end of this module, we will learn how the GPS receivers calculate its own position by minimize the modeled errors based on its measured distances from 4 – 6 GPS satellites.

Goal and Objectives

We will the study optimization method of multivariate functions under zero to two constraint functions in the setting of several different applications. The objectives are:

1. Review the concepts of objective functions and constraints
2. Lagrange Multiplier methods with a single constraints
3. Lagrange Multiplier methods with multiple constraints
4. Use MATLAB to find optimal solutions

Reflection Questions

The difference between this module and your calculus is that we discuss more realistic setting in higher dimensional spaces and multiple constraint equations. If you can connect the ideas and concepts to what you learned in calculus, you will not feel intimidated by the complexity of the symbolic notations. Before you start this lesson, think these three questions.

1. Can you identify the similarity between a parabola of one variable and a circular paraboloid of two variables, between a cubic power function in one variable and a saddle surface in two variables?
2. How do you find critical points in differential calculus and what is the relationship between critical points and local extreme points in theory and in practices?
3. Have you heard of gradient descent/ascent algorithms or hill climbing algorithms? If not, Google it and think about designing your own robotic routing algorithm. How your robots can climb up the hill or get down from a hill efficiently based on touch or Radar sensors?
1. Optimization for objective functions without constraints

Basic Idea
Suppose we have a function $f(x, y)$ or $f(x, y, z)$ defined in some domain and the function has continuous first and second derivatives near its critical point in the domain, we seek a global maximum for it in the domain. The procedure is similar to what we learned in single variable calculus. We need to find the critical points in the interior of the domain first. Then, we compare the values of the function at those critical points with the values of the function at its boundaries to find the maximum.

Definition of Critical Points
The condition for a point $P_0$ to be critical for $f$ is that all directional derivatives of $f$ vanish at $P = P_0$. This is the statement that $\nabla f$ is the zero vector, and all of its components vanish for $(x, y, z) = P(x_0, y_0, z_0)$. That is,

$$\nabla f(P_0) = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right](P_0) = [f_x, f_y, f_z](P_0) = [0, 0, 0], \quad (1.1)$$

Necessary condition: Critical points
Theorem 1, if function $f$ attains its local maximum (or minimum) a point $P_0$, that is

$$f(p_0) \geq f(P) \quad \text{(or } f(p_0) \leq f(P)\text{)} \quad (1.2)$$

for any point $p$ in the neighborhood of point $P_0$, then, $P_0$ must be a critical point of the function $f$.

Connection between critical points and local extreme points
Regardless whether it is single or multivariate, Taylor theorem in last lesson shows that a differentiable function always looks like a quadratic function near its critical point unless all its second derivatives also vanish at the critical point. In such a case, the local behavior of the function degenerates to a plane. The graph of a single variable quadratic function is always a parabola, which either attains its maximum or minimum at the critical point depending on whether the parabola is face down or up. However, the graphs of a multivariate quadratic function are slightly more complicated than that of a single variable quadratic function. Hence, the connection between critical points and maxima and minima is more complicated here than in that in single variable calculus. For a bivariate function, the surface in the neighborhood of a critical point can either look like a paraboloid (like satellite dish) shown in figure 1, or like a saddle surface shown in figure 2.

Graphs of Maximum Minimum and Saddle Points

Figure 1

Figure 2
Examine the extreme attributes (maximum, minimum or neither) of the following 5 bivariate quadratic functions at their common critical point – the origin (0, 0).

(a) \(x^2 + y^2\)
(b) \(-x^2 - y^2\)
(c) \(x^2 - y^2\)
(d) \(xy\)

We may either use MATLAB or manually sketch their graphs near the origin. The answer is:

(a) The function attains its minimum at the critical point because the function is 0 at the origin and positive elsewhere.

(b) The function attains its maximum at the critical point because the function is 0 at the origin and is negative elsewhere.

(c) The origin is a saddle point like the figure 2 for the curve at YZ plane is a parabola of function \(x\) with its face upward and the XZ plane is a parabola of function \(y\) with its face downward. hence it attains neither maximum nor minimum.

(d) both functions attach neither maximum nor minimum at the origin for the same reason as the answer in problem (C).

The four examples (a), (b), (c) and (d) are the four atomic patterns of bivariate quadratic functions near their critical points for all other quadratic functions are modifications of the three atomic patterns with difference scale and orientations.

Notations:
\[A = f_{xx} = \frac{\partial^2 f}{\partial x^2}, \quad B = f_{xy} = \frac{\partial^2 f}{\partial x \partial y}, \quad C = f_{yy} = \frac{\partial^2 f}{\partial y^2}\]

The coefficients of the quadratic that \(f\) resembles at \(P_0\) are determined by the second partial derivatives of \(f\) at \(P_0\). In order for the critical point to be a minimum, the second partials with respect to \(x\) and \(y\) must both be positive, and they must be large enough to dominate the \(xy\) term corresponding to the cross partial \(f_{xy}|P_0\).

Sufficient condition:
The actual condition is the familiar discriminant, \(b^2 - 4ac\), of the quadratic must be negative, which means, in terms of derivatives, that the square of the cross partial is less than the product of the other two:

\[D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} > 0 \text{ or } D = AC - B^2 = f_{xx}f_{yy} - f_{xy}^2 > 0 \quad (1.3)\]

In three dimensions a critical point will be a minimum when the "diagonal partials" are positive, the two dimensional condition holds for all pairs of variables \((x, y), (x, z)\) and \((y, z)\), and the three dimensional determinant of the second partial derivatives is also positive. Table 1 summarizes the characteristics of local maximum and minimum for 2D functions.
Table 1

<table>
<thead>
<tr>
<th>Table for testing local maximum or minimum</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>D</td>
</tr>
<tr>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>-</td>
<td>+</td>
</tr>
<tr>
<td>+/-</td>
<td>-</td>
</tr>
<tr>
<td>+/-</td>
<td>0</td>
</tr>
</tbody>
</table>

Example 2

Classify the attributes of critical points

Find and classify all the critical points of \( x^3 + y^3 - 3xy \)

Solution:

We first need all the first order (to find the critical points) and second order (to classify the critical points) partial derivatives so let’s get those.

\[
\begin{align*}
f_x &= 3x^2 - 3y, \\
f_y &= 3y^2 - 3x, \\
f_{xx} &= 6x, \\
f_{xy} &= 6y, \\
f_{yy} &= -3
\end{align*}
\]

Critical points will be solutions to the system of equations,

\[
\begin{align*}
f_x &= 3x^2 - 3y = 0, \\
f_y &= 3y^2 - 3x = 0
\end{align*}
\]

So, we get two critical points \( P_1(0,0), P_2(1,1) \). All we need to do now is classify them. To do this we will need \( D \). Here is the general formula for \( D \).

\[
D = f_{xx}f_{yy} - f_{xy}^2 = (6x)(6y) - (-3)^2 = 36xy - 9
\]

To classify the critical points all that we need to do is plug in the critical points to the table 1 and use the fact above to classify them.

\[
\begin{array}{c|c|c}
A & D & f \text{ at } P = P_1(0,0) \\
\hline
0 & -9 & \text{Saddle} \\
\end{array}
\]

\[
\begin{array}{c|c|c}
A & D & f \text{ at } P = P_2(1,1) \\
\hline
3 & 27 & \text{Minimum} \\
\end{array}
\]

Figure 3

We get a maximum when all diagonal second derivatives are negative, as is the three by three determinant of second partials, and the two by two determinants are all positive.

Remark: If we change the sign of \( f \) and applying the minimum conditions to \(-f\), the two by two conditions are unaffected by the sign change.
Self-Check Exercises

1. Find and classify all the critical points for

\[ f(x, y) = y^3 + 3x^2y - 3x^2 - 3y^2 + 4 \]

2. If the domain of the function is constrained for \(-1 \leq x \leq 3, -1 \leq y \leq 3\), find the global maximum and minimum by including the points on the boundary.

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2. Lagrange Multiplier methods with a single constraints

Joseph Louis Lagrange (1736-1813) is remembered for his contribution to multivariable calculus and Optimization. He succeeded Euler as the director of Berlin Academy of Germany 1766.

Lagrange used his multiplier method investigating the motion of a particle in space that is constrained to move on a surface defined by an equation \( g(x, y, z) = 0 \). The objective of the investigation is to maximize or minimize a function \( f(x, y, z) \).

Objective function and Constraint equation(s)

Suppose we want to find the minimum value of a function \( f(x, y) \), subject to the condition, \( g(x, y) = 0 \). For this minimum to occur at the point \( p = (x_0, y_0) \), \( p \) must obey that condition, and \( \nabla f \) and \( \nabla g \) must have the same direction at \( p \), i.e. \( \nabla f = \lambda \nabla g \).

This method and its generalizations to higher dimensions, are called the method of Lagrange Multipliers, since it was invented by Lagrange.

\( \nabla f = \lambda \nabla g \) can be written explicitly as

\[ f_x = \lambda g_x, \quad f_y = \lambda g_y \quad (2.1) \]

and \( \lambda \) is called the Lagrange multiplier.

Example 3, Application to optimal dome design

Assume that you need to construct a tank consisting of a right circular cylinder with height \( h \) and radius \( r \), topped with a hemispherical top, and with a flat base, as shown in the figure. If the material for the hemispherical top costs $20/m^2$, and the material for the cylindrical sides costs $8/m^2$, and the material for the circular bottom costs $5/m^2$, find the value of \( r \) and \( h \) that minimize the cost of the materials for this tank, assuming the volume must be 200m³.
Solution: Let $V(r,h)$ and $C(r, h)$ be the volume of the tank and the cost of to build it, respectively, in terms of the radius and height of the cylinder $h$. Then the constraint function is:

$$V(r,h) = \frac{2\pi r^3}{3} + \pi r^2 h = 200\pi \quad (2.2)$$

And the objective function is the minimize

$$C(r, h) = 20 \times 2\pi r^2 + 8 \times 2\pi rh + 5 \times \pi r^2 = \pi(r(45r + 16h)) \quad (2.3)$$

By equation (3.2) we have

$$\frac{\partial V(r,h)}{\partial r} = 2\pi r^2 + 2\pi rh, \quad \frac{\partial V(r,h)}{\partial h} = \pi r^2 \quad (2.4)$$

By equation (3.3) we have,

$$\frac{\partial C(r,h)}{\partial r} = \pi(45r + 16h) + 45\pi r = 90\pi r + 16\pi h, \quad \frac{\partial C(r,h)}{\partial h} = 16\pi r \quad (2.5)$$

Apply formula (3.1) to (3.4) and (3.5)

$$2\pi r^2 + 2\pi rh = \lambda(90\pi r + 16\pi h) \quad (2.6)$$

$$\pi r^2 = \lambda \times 16\pi r \quad (2.7)$$

$\lambda$ is the Lagrange multiplier.

We can obtain $\lambda = r/16$, plug to the equation (3.6)

$$2\pi r^2 + 2\pi rh = \frac{r}{16}(90\pi r + 16\pi h), \quad \longrightarrow \quad h = \frac{29}{8}r \quad (2.8)$$

Plug (3.8) to constraint (3.2), we will find

$$r = \frac{1}{3}\sqrt{4800/103} \text{ m}, \quad h = \frac{29}{8} \sqrt{4800/103} \text{ m}$$

The nice result is that the ratio of the height over the radius is $29/8$.

**Example 4:**

**Informatics Application**

In data mining and statistics, we wish to find the discrete probability distribution on the points $\{x_1, x_2, ..., x_n\}$ with maximal information entropy. This is the same as saying that we wish to find the least (most) biased probability distribution on the points $\{x_1, x_2, ..., x_n\}$. In other words, we wish to maximize the Shannon entropy equation:

$$f(p_1, p_2, ..., p_n) = -\sum_{j=1}^{n} p_j \log_2 p_j \quad (2.9)$$
For this to be a probability distribution the sum of the probabilities \( p_j \) at each point \( x_j \) must equal 1, so our constraint is

\[
g(p_1, p_2, \ldots, p_n) = \sum_{j=1}^{n} p_j = 1 \tag{2.10}
\]

We use Lagrange multipliers to find the point of maximum entropy.

Solution: Take the partial derivatives to the equations of (2.9) and (2.10),

\[
\frac{\partial f}{\partial p_k} = -\log_2 p_k - \frac{1}{\ln2}, \quad \frac{\partial g}{\partial p_k} = 1, k = 1, 2, \ldots n ,
\]

Apply the Lagrange multiplier formula, we have a system of \( n \) equations

\[
-\log_2 p_k - \frac{1}{\ln2} = \lambda, \quad k = 1, 2, \ldots n
\]

which gives that all \( \log_2 p_k \) are equal. By using the constrain (2.10), we find \( p_k = 1/n \). Hence, the uniform distribution is the distribution with the greatest entropy, among distributions on \( n \) points.

Self-Check Exercises

3. find the maximum volume of the box without top lid that you can make from a rectangular card board of size \( 144 \text{ ft}^2 \) by cutting off the 4 corners.

4. Use Lagrange Method to find the maximum and minimum of

\[
f(x, y, z) = xyz \text{ under the constraint } x^2 + 4y^2 + 9z^2 = 36
\]

3. Lagrange multiplier methods with multiple constraints

Necessary conditions

Suppose we are constrained to move on a curve in 3 dimensions: we want to find the critical points for function \( f(x, y, z) \), given

\[
g(x, y, z) = 0 \text{ and } h(x, y, z) = 0. \quad (3.1)
\]

The condition we want to impose is that \( \nabla f \) has no component tangent to the curve, which means that \( \nabla f \) lies in the plane of and \( \nabla g \) and \( \nabla h \). This means that \( \nabla f \) can be written as \( \lambda \nabla g + \mu \nabla h \). This statements amounts to three equations, one for each vector component, with two new unknowns, \( \lambda \) and \( \mu \).

\[
\nabla f = \lambda \nabla g + \mu \nabla h \quad (3.2)
\]

Example 5

Multiple Constraint Equations

Maximize \( x_1y_1 + x_2y_2 \) with two constraints \( x_1^2 + y_1^2 = 1, \ x_2^2 + y_2^2 = 1 \)

Solution:

\[
g = x_1^2 + y_1^2 - 1 = 0 \quad h = x_2^2 + y_2^2 - 1 = 0 \quad f = x_1y_1 + x_2y_2 \quad \nabla f = \lambda \nabla g + \mu \nabla h
\]

Observe the symmetry of \( x_1, y_1, x_2 \) and \( y_2 \), we will easily find the
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critical points as when
\[ x_1 = y_1 = \pm \frac{\sqrt{2}}{2}, \quad x_2 = y_2 = \pm \frac{\sqrt{2}}{2}. \]
Hence the maximum of \( x_1y_1 + x_2y_2 = 1 \).

Alternative algebraic proof:
\[ x_1y_1 \leq \frac{x_1^2 + y_1^2}{2} \leq \frac{1}{2}, \quad x_2y_2 \leq \frac{x_2^2 + y_2^2}{2} \leq \frac{1}{2} \]
Hence, \( x_1y_1 + x_2y_2 \leq 1 \) and the equality holds when all 4 variables equal each other.

Find the maximum and minimum volumes of a rectangular box whose surface area is 1500 square cm, and whose total edge length is 200 cm.

Solution: Then the constraint function and the objective functions are
\[ S = g = 2(xy + xz + yz) - 1500 = 0 \quad (1) \]
\[ D = h = 4(x + y + z) - 200 = 0 \quad (2) \]

From the equation (3.2), we have
\[ yz = 2(y + z)\lambda + 4\mu \quad (3) \]
\[ xz = 2(x + z)\lambda + 4\mu \quad (4) \]
\[ xy = 2(x + y)\lambda + 4\mu \quad (5) \]
\[ (x - y)(z - 2\lambda) = 0 \quad (4) - (3) \]
\[ (x - z)(y - 2\lambda) = 0 \quad (5) - (3) \]
\[ (y - z)(x - 2\lambda) = 0 \quad (5) - (4) \]
In either choice, we end up \( x = y = z \) as we expected. Since it is impossible for the two constraints to hold equality, we need to figure out which constraint is more restrictive. If we put \( x = y = z \) to (2), we have \( x = y = z = 50/3 \) and find \( S = 2(xy + xz + yz) = 6 \times (50/3)^2 = 5000/3 > 1500 \).

Hence this solution does not confirm to the constraint (1). And the constraint (1) is more restrictive. We put \( x = y = z \) to (1), we have \( 6x^2 = 1500, x^2 = 250, x = y = z = 5\sqrt{10} \). Now we check the constraint (2).
\[ 4(x + y + z) = 60\sqrt{10} = 189.7366596 < 200. \] Hence, \( x = y = z = 5\sqrt{10} \) gives us the maximum volume \( V = xyz = 3952.847 \).

Example 7

Use MATLAB to solve algebraic equations

4. MATLAB commands

The solve command is used to find solutions of equations involving symbolic expressions.

```matlab
>> solve('sin(x)+x=5')
ans =
```

Mathematical Modeling and Simulation, Module 2: Matrix Calculus and Optimization

Unit 1: Introduction to Linear Regression
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5.6175550052726989176213921571114

In expressions with more than one variable, we can solve for one or more of the variables in terms of the others. Here we find the roots of the quadratic $ax^2+bx+c$ in $x$ in terms of $a$, $b$ and $c$. By default solve sets the given expression equal to zero if an equation is not given.

```
>> solve('a*x^2+b*x+c','x')
ans =
  [1/2/a*(-b+(b^2-4*a*c)^(1/2))]
  [1/2/a*(-b-(b^2-4*a*c)^(1/2))]
```

Systems of equations can also be handled by `solve`.

```
>> S=solve('x+y+z=1','x+2*y-z=3')
S =
x: [1x1 sym]
y: [1x1 sym]
```

The variable $S$ contains the solution, which consists of $x$ and $y$ in terms of $z$.

```
>> S.x
ans =
-3*z-1
>> S.y
ans =
2*z+2
```

Now let's find the points of intersection of the circles $x^2+y^2=4$ and $(x-1)^2+(y-1)^2=1$.

```
>> S=solve('x^2+y^2=4','(x-1)^2+(y-1)^2=1')
S =
x: [2x1 sym]
y: [2x1 sym]
```

The variable $S$ contains the solution, which consists of $x$ and $y$ in terms of $z$. We will solve a Lagrange multiplier problem. For $f(x,y)=xy(1+y)$ let's find the maximum and minimum of $f$ on the unit circle $x^2+y^2=1$. First we enter the function $f$ and the constraint function $g(x,y)=x^2+y^2-1$.

```
>> syms x y mu
>> f=x*y*(1+y)
f =
x*y*(1+y)
>> g=x^2+y^2-1
g =
x^2+y^2-1
```

Next we solve the Lagrange multiplier equations (2.1) and constraint equation $g(x,y)=0$ for $x$, $y$ and $m$.

```
>> L=jacobian(f)-mu*jacobian(g)
L = [ y*(1+y)-2*mu*x, x*(1+y)+x*y-2*mu*y]
>> S=solve(L(1),L(2),g)
S =
  mu: [5x1 sym]
```
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\[ x: \begin{bmatrix} 5x1 \text{ sym} \end{bmatrix} \]
\[ y: \begin{bmatrix} 5x1 \text{ sym} \end{bmatrix} \]

Next let's view the critical points found. We can ignore \( m \) now.

\[
>> \begin{bmatrix} S.x \ S.y \end{bmatrix} \\
\text{ans} = \\
\begin{bmatrix}
\frac{1}{6} \times (22 - 2 \times 13^{1/2})^{1/2}, & \frac{1}{6} + \frac{1}{6} \times 13^{1/2} \\
-\frac{1}{6} \times (22 - 2 \times 13^{1/2})^{1/2}, & \frac{1}{6} + \frac{1}{6} \times 13^{1/2} \\
\frac{1}{6} \times (22 + 2 \times 13^{1/2})^{1/2}, & \frac{1}{6} - \frac{1}{6} \times 13^{1/2} \\
-\frac{1}{6} \times (22 + 2 \times 13^{1/2})^{1/2}, & \frac{1}{6} - \frac{1}{6} \times 13^{1/2} \\
0, & -1
\end{bmatrix}
\]

Next we need to evaluate \( f \) at each of these points.

\[
>> \text{values} = \text{simple(subs(f,} \{x,y\},\{S.x,S.y\}) \\
\text{values} = \\
\begin{bmatrix}
\frac{1}{216} \times (22 - 2 \times 13^{1/2})^{1/2} \times (1 + 13^{1/2}) \times (7 + 13^{1/2}) \\
-\frac{1}{216} \times (22 - 2 \times 13^{1/2})^{1/2} \times (1 + 13^{1/2}) \times (7 + 13^{1/2}) \\
\frac{1}{216} \times (22 + 2 \times 13^{1/2})^{1/2} \times (-1 + 13^{1/2}) \times (-7 + 13^{1/2}) \\
-\frac{1}{216} \times (22 + 2 \times 13^{1/2})^{1/2} \times (-1 + 13^{1/2}) \times (-7 + 13^{1/2}) \\
0
\end{bmatrix}
\]

Finally we convert these into decimal expressions to identify the maximum and minimum. This is done using the double command.

\[
>> \text{double(values)} \\
\text{informatics} = \\
0.8696 \\
-0.8696 \\
-0.2213 \\
0.2213 \\
0
\]

Thus the maximum off is about 0.8696 and the minimum is about -0.8696.

**Review Exercises**

1. Find the closest point in quadrant I on the curve implicitly determined by \( x^2 + xy = 1 \) to the origin (hint, you can use the square of the distance as your objective function to avoid the square root of the distance function).

2. Guess what is the solution above to obtain the minimum information entropy, i.e. information gain in the example 5 above, then justify your answers.

3. Maximize \( x_1 y_1 + x_2 y_2 + x_3 y_3 \) with three constraints
\[
x_1^2 + y_1^2 = 1, \quad x_2^2 + y_2^2 = 1, \quad x_3^2 + y_3^2 = 1
\]

4. Try to use the MATLAB to find the maximum and minimum volumes of a rectangular box whose surface area is 1500 square cm.

5. Find the maximum and minimum volumes of a rectangular box whose total edge length is 200 cm.
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Answers to Self-Check Exercises

1. Solution:

\[ \nabla f = [6xy, -6x, 3y^2 + 3x^2 - 6y] = [0, 0] \]

By the second equation when \( x = 0, y = 0 \) or \( y = 1 \),

When \( y = 1, x = 1 \) or \( x = -1 \),

Hence, there are four critical points \((0, 0), (0, 2), (1, 1), \text{ and } (-1, 1)\).

Taking the second derivatives

\[ f_{xx} = 6y - 6, \quad f_{yy} = 6y - 6, \quad f_{xy} = 6x, \Delta = f_{xx}f_{yy} - f_{xy}^2 = 36(y - 1)^2 - x^2 \]

<table>
<thead>
<tr>
<th>Critical point</th>
<th>determinant ( \Delta )</th>
<th>conclusion</th>
</tr>
</thead>
<tbody>
<tr>
<td>((0, 0))</td>
<td>(\Delta = 36)</td>
<td>-6</td>
</tr>
<tr>
<td>((0, 2))</td>
<td>(\Delta = 36)</td>
<td>6</td>
</tr>
<tr>
<td>((1, 1))</td>
<td>(\Delta = -36)</td>
<td>0</td>
</tr>
<tr>
<td>((-1, 1))</td>
<td>(\Delta = -36)</td>
<td>0</td>
</tr>
</tbody>
</table>

2. We need to compare the internal maximum \( f(0,0) = 4 \) and minimum at \( f(0, 2) = 0 \) with the maximum and minimum occurs on the four boundary line segments.

\[
\begin{align*}
 f(x, -1) &= (-1)^3 + 3x^2(-1) - 3x^2 - 3(-1)^2 + 4 = -6x^2, \quad -1 \leq x \leq 3 \\
 \max_{-1 \leq x \leq 3} f(x, -1) &= 0, \quad \text{and} \quad \min_{-1 \leq x \leq 3} f(x, -1) = -36, \\
 f(x, 3) &= (3)^3 + 3x^2(3) - 3x^2 - 3(3)^2 + 4 = 6x^2 - 14, \quad -1 \leq x \leq 3 \\
 \max_{-1 \leq x \leq 3} f(x, 3) &= 40, \quad \min_{-1 \leq x \leq 3} f(x, 3) = -14, \\
 f(-1, y) &= y^4 + 3(-1)^2y - 3(-1)^2 - 3y^2 + 4 = y^3 - 3y^2 + 3y + 1, \\
 f'(-1, y) &= 3y^2 - 6y + 3 = 3(y - 1)^2, \quad \text{critical point } y = 1, \\
 \max_{-1 \leq y \leq 3} f(-1, y) &= 8, \quad \min_{-1 \leq y \leq 3} f(-1, y) = -6 \\
 f(3, y) &= y^3 + 3(3)^2y - 3(3)^2 - 3y^2 + 4 = y^3 - 3y^2 + 27y - 23, \\
 f'(3, y) &= 3y^2 - 6y + 27 = 3(y - 1)^2 + 14, \quad \text{no critical point} \\
 \max_{-1 \leq y \leq 3} f(3, y) &= 58, y = 3 \quad \min_{-1 \leq y \leq 3} f(3, y) = -54, y = -1 \\
\end{align*}
\]

Hence, it is clear that the maximum occurs at corner point \( f(3,3) = 58 \), minimum also occurs at the corner point at \( f(3, -1) = -54 \)

3. Solution, let the side length of the box be \( x \), width of the box be \( y \) and height be \( z \), the volume \( V = xyz \), the constraints is \( S = (x+2z)(y+2z) = 144 \), the four corners cut off are four squares of side length \( z \).

\[ \nabla V = [yz, xz, xy], \quad \nabla S = [y + 2z, x + 2z, 8z + 2(x + y)] \]

By Lagrange multiplier theorem, we have

\[ [yz, xz, xy] = \lambda[y + 2z, x + 2z, 8z + 2(x + y)] \]

The symmetry of the first two equations about \( x \) and \( y \) reveals that \( x = y \).

By the constraint of \( S = 144 \), we have \( x = y = 12 - 2z \),

\[ V = z(12 - 2z)^2, \quad 0 < z < 6, \quad \text{since } x \text{ and } y \text{ must be positive.} \]
Optimizations and Lagrange Multiplier Method

\[ V' = (12 - 2z)^2 + 2z(-2)(12 - 2z) = 0, \quad (12 - 2z)(12 - 2z) - 4z = 0, z = 2, \text{ or } z = 6 \]

Since \( z = 6 \) makes \( x = y = 0 \), the dimensions does not makes a box. Hence,
\( z = 2, \ x = y = 8 - 4 = 4. \)

The maximum box is \( 2 \times 4 \times 4 = 32ft^3 \) when length and width are both 4 and height 2.

4. Solution: \( \nabla f = [yz, xz, xy], \ \nabla g = [2x, 8y, 18z], \) by Lagrange multiplier theorem

\[ \nabla f = \lambda \nabla g, \ [yz, xz, xy] = \lambda [2x, 8y, 18z] \]

Multiplying \( x, \ y, \) and \( z \) to the three equations, respectively, we have

\[ 2\lambda x^2 = 8\lambda y^2 = 18\lambda z^2 = xyz, \]

If \( \lambda = 0, \) it is easy to see from the equations above that we have \( x = y = z = 0, \)

\( f(x, y, z) = xyz = 0 \)

if \( \lambda \neq 0, \) then \( x^2 = 4y^2 = 9z^2, \)

plug into the constraint equation, \( x^2 + 4y^2 + 9z^2 = 36 \)

We have \( x^2 = 4y^2 = 9z^2 = 12 \), It follows that \( x = \pm 2\sqrt{3}, y = \pm \sqrt{3}, z = 2\sqrt{3}/3 \)

Then, we have the maximum is \( f(x, y, z) = 2\sqrt{3} \ast \sqrt{3} \ast 2\sqrt{3}/3 = 4\sqrt{3}, \)

Minimum \( f(x, y, z) = -2\sqrt{3} \ast \sqrt{3} \ast \frac{2\sqrt{3}}{3} = -4\sqrt{3} \)